All the components of the curvature spinors defined in (73)–(76) are (p,q)-weighted quantities for certain integers p, q. Also most of the spin coefficients are weighted quantities and indeed only α, β, γ, ϵ, and θ are nonweighted.[9] This raises the possibility of an extension of the Geroch, Held, and Penrose (GHP) formalism[10] to dimension 5. In fact, it is not very difficult to introduce the weighted differential operators in terms of the Newman–Penrose frame differentials. One concludes that the weighted differential operators constructed from D, Δ, δ, and $\bar{\delta}$ coincide with the four-dimensional definitions of, respectively, Þ, Þ$'$, $\eth$, $\eth'$ as shown in Ref. 9.

Other interesting issue is the algebraic classification of the Weyl spinor. This has been tackled in Ref. 4 where an invariant classification of this spinor was put forward. Under this classification, there are 12 different "Petrov types" of the Weyl spinor so it would be interesting to find out how one can characterize these Petrov types in terms of conditions involving the components of the Weyl spinor (some cases are already analyzed in Ref. 4). Alternatively, one could try to apply the alignment theory directly to the Weyl spinor and devise a classification for it as in Refs. 14 and 3. This theory is based on studying the boost weights of those scalar components of the Weyl tensor which do not vanish on a suitably chosen frame and hence it is clear that we could follow the same procedure if we used the scalar components of the Weyl spinor and the notion of boost weight discussed above. Indeed, some Petrov types adopt a simpler form when we work with the components of the Weyl spinor. For example, a spacetime is of Petrov type D if and only if the components of the Weyl spinor different from zero are those of boost weight zero. These are

$$\Psi_2, \quad \Psi_{11}, \quad {}^*\Psi_2, \quad \Psi_2^*, \quad \Psi_{02}.$$

One could take this as the starting point of a systematic study of all the possible five-dimensional (vacuum) type D exact solutions of the Einstein equations. To that end, one sets to zero in the five-dimensional Newman–Penrose equations all the curvature scalars except those shown in previous equation (if we do not work in vacuum then we need to retain the components of the Ricci spinor) and then checks the consistency with the commutation relations shown below. Work in this direction has been already started in Ref. 9 for the vacuum case using the extension of the GHP formalism mentioned above.

The five-dimensional spinor calculus is now being implemented in the MATHEMATICA package *Spinors,*[8] which is part of the *xAct* system.[12]

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APPENDIX: SYMPLECTIC METRICS ON A VECTOR SPACE

Let $\mathbf{V}$ and $\mathbf{V}^*$ be, respectively, a vector space (real or complex) and its dual and let us use small Latin characters $a,b,c,\ldots$ to denote the abstract indices of the elements of the tensor algebra built with $\mathbf{V}$ and $\mathbf{V}^*$, which is $\mathfrak{T}(\mathbf{V})$. We introduce next two quantities M_{ab} and T^{ab} establishing linear isomorphisms $M:\mathbf{V}\to\mathbf{V}^*$ and $T:\mathbf{V}^*\to\mathbf{V}$ in the following way:

$$v_a \equiv M_{ab}v^b \quad \text{and} \quad \omega^a \equiv T^{ab}\omega_b \quad \text{for any } v^a \in \mathbf{V}, \quad \omega_a \in \mathbf{V}^*. \tag{A1}$$

Note the convention of having only the second indices of M and T as contracted indices. Previous isomorphisms are generalized to $\mathfrak{T}(\mathbf{V})$ in the obvious way and shall be referred to as the operation of "raising and lowering of indices." In addition, we impose that $T=M^{-1}$ and so (A1) implies

$$T^{ab}M_{bc} = \Delta^a{}_c, \tag{A2}$$

with $\Delta^a{}_c$ the identity on $\mathbf{V}$ (Kronecker delta on $\mathbf{V}$), and

$$M_{ab}T^{bc} = \delta_a{}^c, \tag{A3}$$

with $\delta_a{}^c$ the identity on $\mathbf{V}^*$ (Kronecker delta on $\mathbf{V}^*$).

We can change indices with the Kronecker delta tensors, and now we can also raise and lower indices by making use of the M and T isomorphisms. Suppose now that we wish to compute the product $M_{ab}\Delta^b{}_c$. We can either lower an index of Δ or change an index of M. We conclude

$$M_{ac} = \Delta_{ac} \tag{A4}$$

and, similarly,

$$T^{ac} = \delta^{ac}. \tag{A5}$$

We can also see that

$$T^{ab} = M^{ba} \quad \text{and} \quad T_{ab} = M_{ba}, \tag{A6}$$

independent of the symmetries of M and T, which could even have no symmetry at all. Concluding, we always have, for indices of any character, and any symmetry

$$T^{ab} = M^{ba} = \Delta^{ba} = \delta^{ab}, \quad T_{ab} = M_{ba} = \Delta_{ba} = \delta_{ab}, \quad T_a{}^b = \delta_a{}^b, \quad M^a{}_b = \Delta^a{}_b. \tag{A7}$$

The four quantities T, M, Δ, and δ are essentially the same. Let us take, for clarity, only T. It always obeys

$$v^a = T^{ab}v_b, \quad v_a = T_a{}^b v_b, \quad v_b = v^a T_{ab}, \quad v^b = v^a T_a{}^b. \tag{A8}$$

However, the following are generically undefined

$$T_{ab}v^b, \quad T^a{}_b v^b, \quad v_a T^{ab}, \quad v_a T^a{}_b, \tag{A9}$$

unless T^{ab} has a definite symmetry which means that either T^{ab} is symmetric or antisymmetric. When this is the case, we deduce from (A7) that T_{ab} M^{ab}, M_{ab}, Δ_{ab}, δ_{ab}, Δ^{ab}, and δ^{ab} all inherit the symmetry of T_{ab} and indeed we could just regard the quantity T^{ab} as fundamental and the remaining ones as derived from it, keeping the symbol T as the kernel letter for all of them. Also using (A7), one may deduce

$$\delta_a{}^b = \Delta^b{}_a, \tag{A10}$$

if T^{ab} is symmetric and

$$\delta_a{}^b = -\Delta^b{}_a,$$

if T^{ab} is antisymmetric. In the case of T^{ab} being symmetric then one introduces a quantity δ^a_b to mean either $\delta_a{}^b$ or $\Delta^b{}_a$ and no confusion can arise. However, if T^{ab} is antisymmetric and we insist on keeping only one delta symbol δ^a_b, we need to specify also whether δ^a_b refers to $\delta_a{}^b$ or to $\Delta^b{}_a$. We believe that to keep the notation δ^a_b in this context is somewhat confusing and one should instead pick up one of the "deltas" as the fundamental one and regard the other as a derived quantity. For example, if we agree to take $\delta_a{}^b$ as the fundamental quantity (as we do in our discussion in Sec. II), then we have

$$\delta^a{}_b = \Delta^a{}_b = -\delta_b{}^a,$$

and no confusion arises.

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